Twitter Thread by 10-K Diver

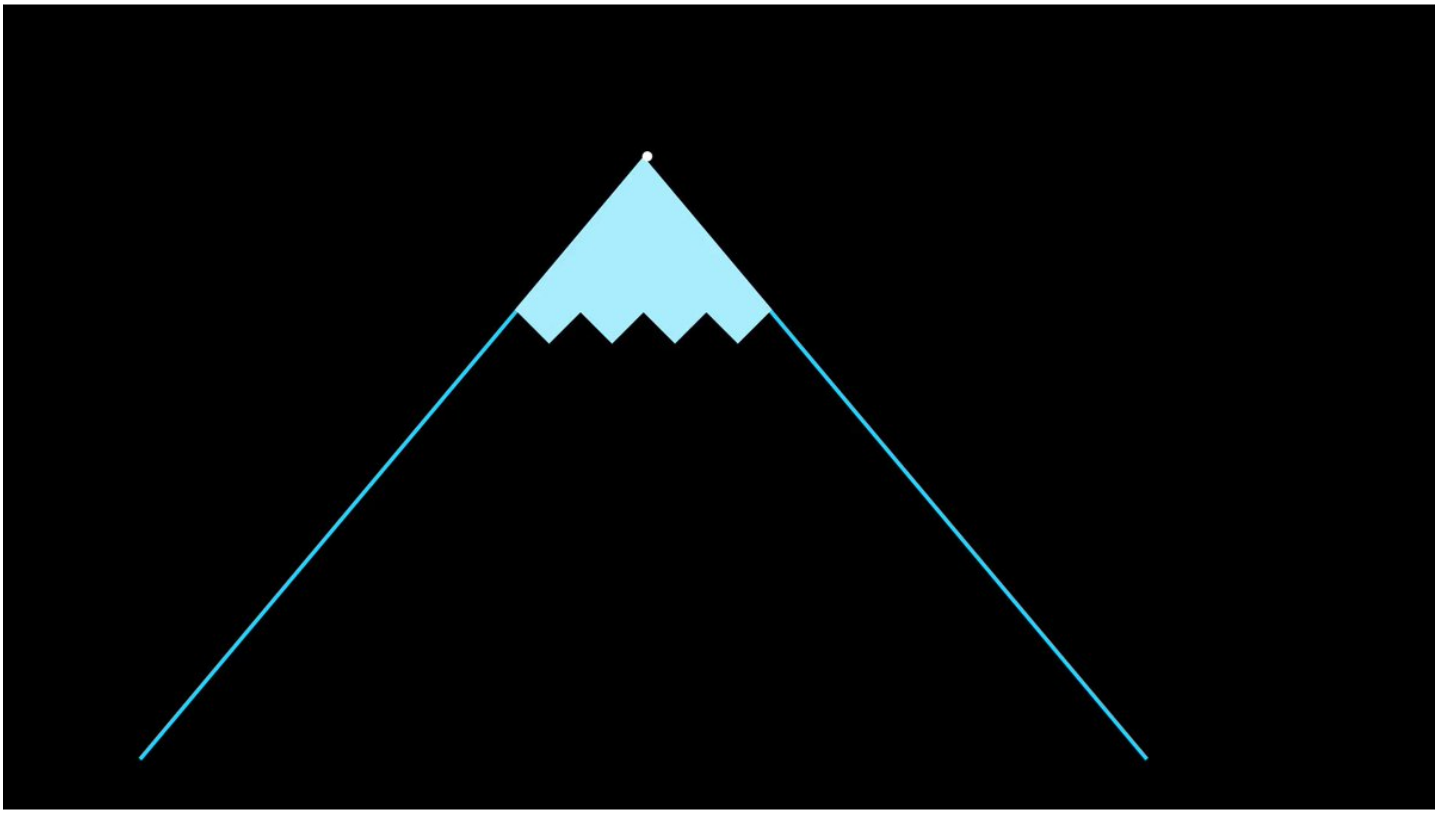
10-K Diver
@10kdiver

1/

Get a cup of coffee.

In this thread, let's talk snowballs.

Snowballs are super fun! And they can teach us so much about life, about things that grow over time, their rates of growth, compounding, etc.

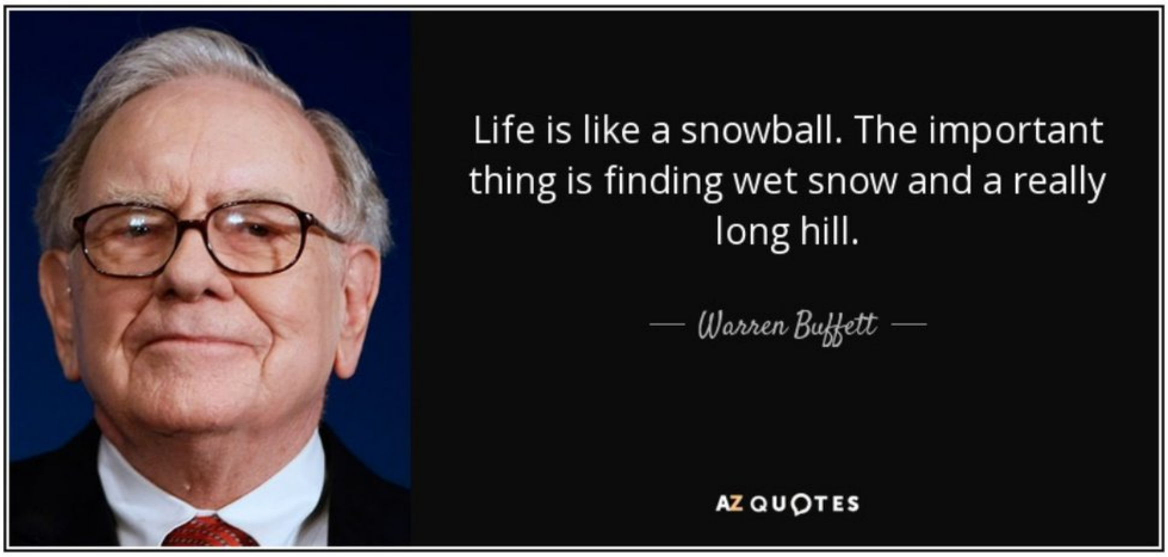


2/

Snowballs are often used as a metaphor for compounding.

A snowball starts small at the top of a hill. As it rolls downhill, it picks up speed and grows in size. This is like money compounding over time.

For example, here's Buffett's famous "snowball quote":

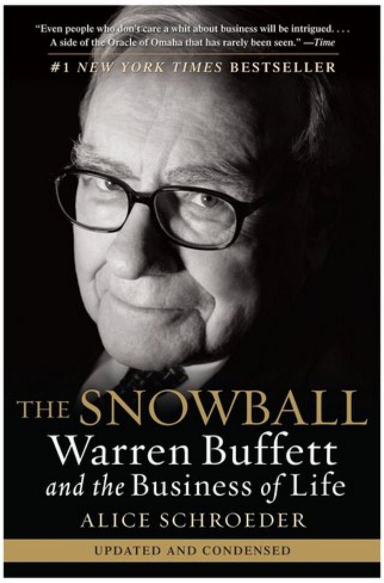


3/

There's even a famous book about Buffett with "snowball" in the title.

The book's theme is similar to the quote above: the process of compounding is like a snowball that grows over time as it rolls downhill.

Link: https://t.co/L3opOrdeoZ



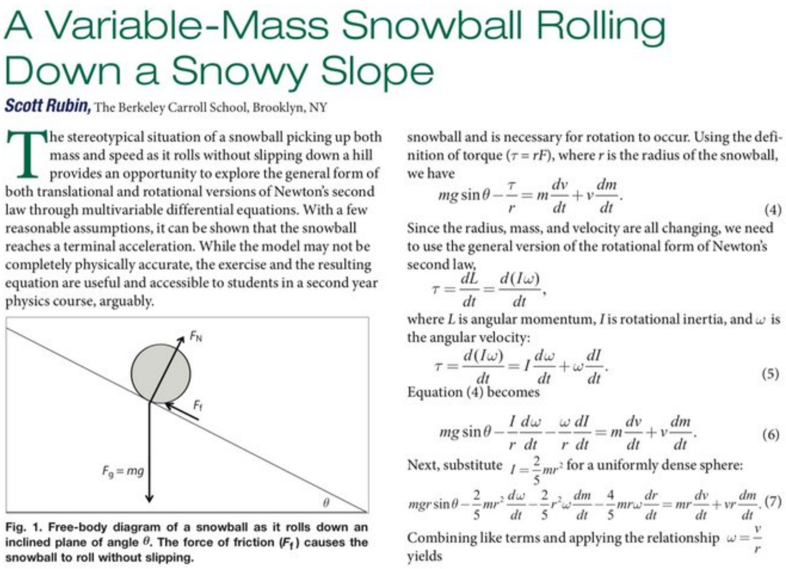
4/

Clearly, snowballs rolling downhill are worthy objects of study.

So let's dive into their physics!

Luckily for us, in 2019, Scott Rubin published a paper analyzing such snowballs -- in a journal called "The Physics Teacher".

All we need to do is understand this paper.



A Variable-Mass Snowball Rolling Down a Snowy Slope

Scott Rubin, The Berkeley Carroll School, Brooklyn, NY

The stereotypical situation of a snowball picking up both mass and speed as it rolls without slipping down a hill provides an opportunity to explore the general form of both translational and rotational versions of Newton's second law through multivariable differential equations. With a few reasonable assumptions, it can be shown that the snowball reaches a terminal acceleration. While the model may not be completely physically accurate, the exercise and the resulting equation are useful and accessible to students in a second year physics course, arguably.

Fig. 1. Free-body diagram of a snowball as it rolls down an inclined plane of angle θ. The force of friction (F_f) causes the snowball to roll without slipping.

snowball and is necessary for rotation to occur. Using the definition of torque ($\tau = rF$), where r is the radius of the snowball, we have

$$mg\sin\theta - \frac{\tau}{r} = m\frac{dv}{dt} + v\frac{dm}{dt}. \quad (4)$$

Since the radius, mass, and velocity are all changing, we need to use the general version of the rotational form of Newton's second law,

$$\tau = \frac{dL}{dt} = \frac{d(I\omega)}{dt},$$

where L is angular momentum, I is rotational inertia, and ω is the angular velocity:

$$\tau = \frac{d(I\omega)}{dt} = I\frac{d\omega}{dt} + \omega\frac{dI}{dt}. \quad (5)$$

Equation (4) becomes

$$mg\sin\theta - \frac{I}{r}\frac{d\omega}{dt} - \frac{\omega}{r}\frac{dI}{dt} = m\frac{dv}{dt} + v\frac{dm}{dt}. \quad (6)$$

Next, substitute $I = \frac{2}{5}mr^2$ for a uniformly dense sphere:

$$mgr\sin\theta - \frac{2}{5}mr^2\frac{d\omega}{dt} - \frac{2}{5}r^2\omega\frac{dm}{dt} - \frac{4}{5}mr\omega\frac{dr}{dt} = mr\frac{dv}{dt} + vr\frac{dm}{dt}. \quad (7)$$

Combining like terms and applying the relationship $\omega = \frac{v}{r}$ yields

5/

We begin by identifying 2 kinds of quantities in our "snowball system":

1. "Parameters" that don't change with time (eg, the hill's angle of incline), and

2. "State Variables" that *do* change with time (eg, the snowball's radius and velocity).

Quantities in our snowball system

1. Parameters that don't change with time :

$$\vec{P} = \begin{bmatrix} \theta \\ g \\ \rho \\ k \end{bmatrix} \begin{array}{l} \to \text{angle of incline of the hill} \\ \to \text{acceleration due to gravity } (\sim 9.81 \, m/s^2) \\ \to \text{density of snow } (\sim 100-300 \, kg/m^3) \\ \to \text{increase in snowball's radius per complete rotation} \end{array}$$

2. State Variables that change with time:

$$\vec{x} = \begin{bmatrix} r \\ v \end{bmatrix} \begin{array}{l} \to \text{radius of snowball} \\ \to \text{velocity of snowball as it rolls down the hill} \end{array}$$

6/

We then derive other useful quantities from our parameters and state variables.

Such "derived" quantities include our snowball's mass, its moment of inertia, angular momentum, etc.

This all follows from basic math and physics (eg, the formula for the volume of a sphere).

Formulas for various snowball variables

* Mass m of snowball $= \begin{pmatrix}\text{Density of} \\ \text{snow}\end{pmatrix} * \begin{pmatrix}\text{Volume of} \\ \text{snowball}\end{pmatrix}$

$$= \rho * \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \rho r^3$$

* Angular velocity $\omega = \dfrac{v}{r}$

* Momentum $p = m * v = \dfrac{4}{3}\pi \rho r^3 v$

* Moment of Inertia $I = \dfrac{2}{5} m r^2$

$$= \frac{2}{5} \cdot \frac{4}{3}\pi \rho r^3 \cdot r^2$$

$$= \frac{8}{15}\pi \rho r^5$$

* Angular momentum $L = I\omega$

$$= \frac{8}{15}\pi \rho r^5 \cdot \frac{v}{r}$$

$$= \frac{8}{15}\pi \rho r^4 v$$

7/

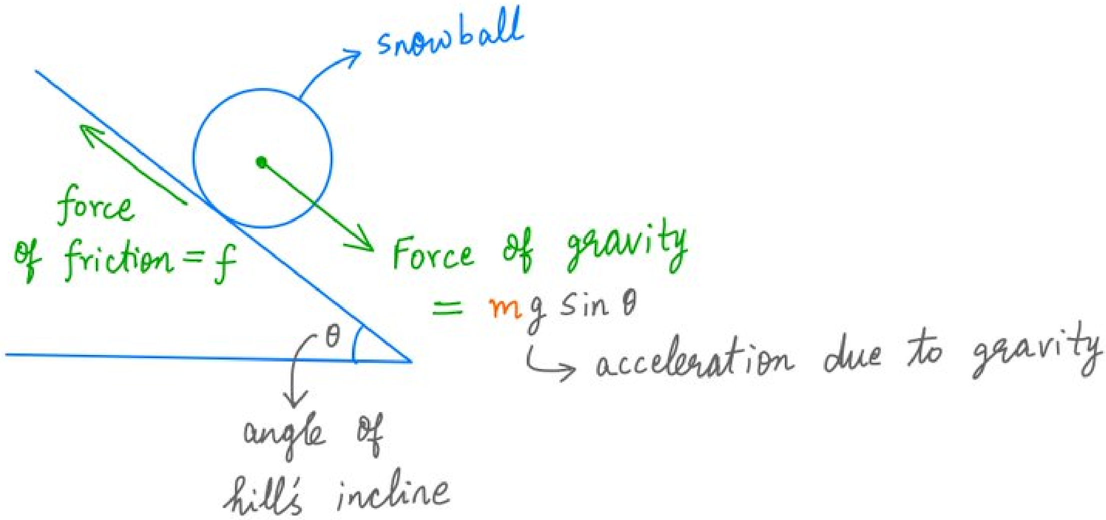
Then we write the "laws of motion" for our snowball.

These are based on various physics principles -- like how a system behaves when subject to torque, Newton's second law, how our snowball accumulates snow as it rolls, etc.

Note: this requires some knowledge of calculus.

Equations from the Laws of Physics

Torque Law



Rate of change of angular momentum = Torque from friction

$$\Rightarrow \quad \dot{L} = f \cdot r$$

But $L = \frac{8}{15}\pi\rho r^4 \vartheta$.

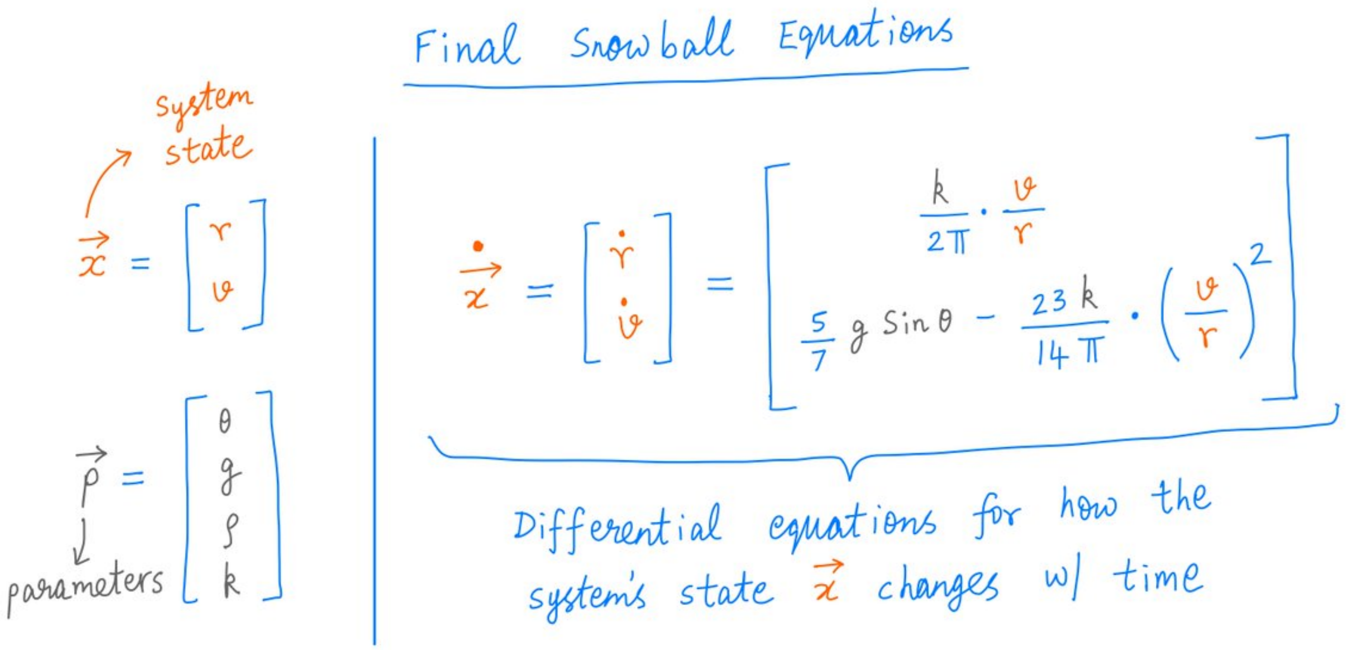
$$\Rightarrow \quad \frac{8}{15}\pi\rho\left(r^4\dot{\vartheta} + 4r^3\dot{r}\vartheta\right) = f \cdot r$$

$$\Rightarrow \quad f = \frac{8}{15}\pi\rho r^2\left(r\dot{\vartheta} + 4\dot{r}\vartheta\right)$$

8/

Finally, we tie everything together by creating a system of "differential equations".

These equations describe how our snowball's radius and velocity evolve over time -- as it rolls downhill.



$$\vec{x} = \begin{bmatrix} r \\ v \end{bmatrix} \qquad \vec{p} = \begin{bmatrix} \theta \\ g \\ \rho \\ k \end{bmatrix}$$

$$\dot{\vec{x}} = \begin{bmatrix} \dot{r} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} \frac{k}{2\pi} \cdot \frac{v}{r} \\ \frac{5}{7} g \sin\theta - \frac{23k}{14\pi} \cdot \left(\frac{v}{r}\right)^2 \end{bmatrix}$$

9/

The beauty of our differential equations is:

Given our snowball's state (ie, its radius and velocity) at any *one* time, our differential equations allow us to predict its state at any *future* time.

10/

All we need to know is the snowball's initial radius -- when it's at the top of the hill and just starting to roll down.

Just from this, we can calculate our snowball's entire trajectory -- its radius, mass, velocity, momentum, etc., at *every* point on its journey.

11/

How exactly do we calculate all this?

Well, there are standard algorithms to simulate differential equations on a computer.

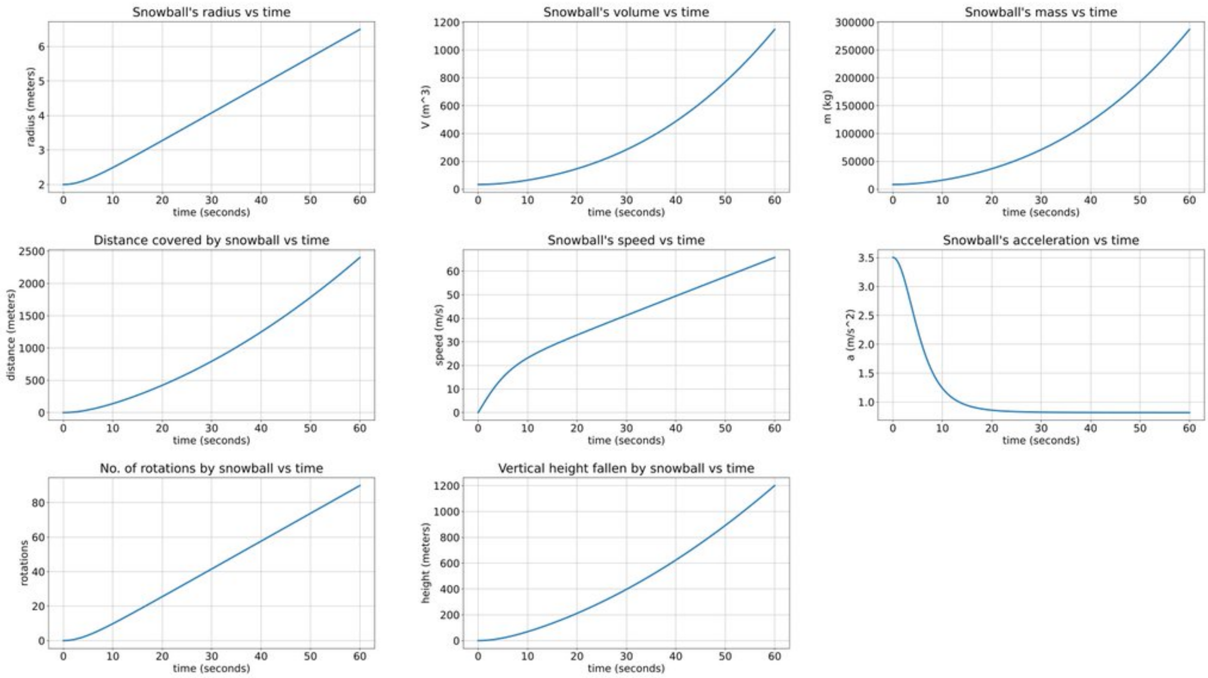
And our snowball's differential equations are fairly simple. So it's not hard to write a program that simulates a snowball rolling downhill.

12/

In fact, I've written such a program.

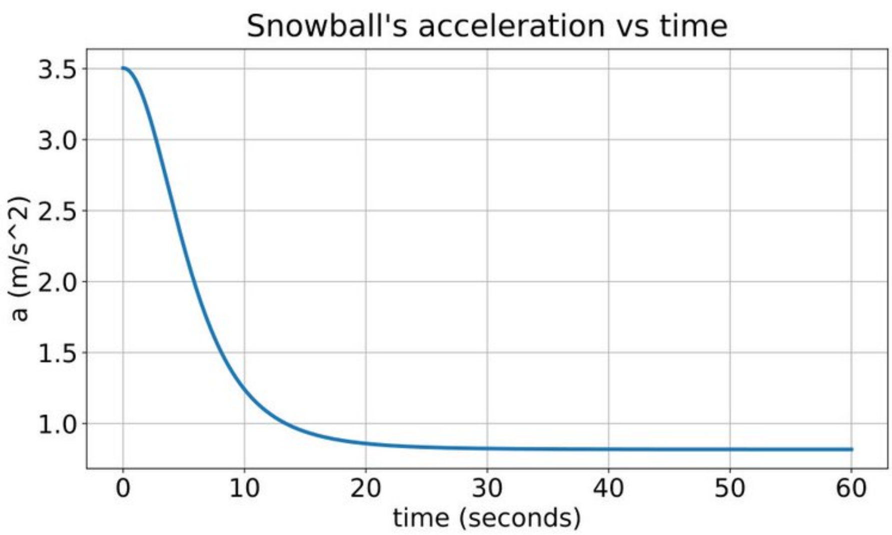
That's how I created the snowball GIF at the top of this thread -- by simulating the differential equations describing our snowball.

Here are some sample plots produced by this program:



13/

The key thing to note here is that the snowball's acceleration -- ie, the rate at which its speed increases -- seems to level off with time:



14/

The paper above by Scott Rubin demonstrates that this must hold true for all snowballs obeying our differential equations: their accelerations must eventually go flat.

And that's a problem -- because it contradicts our nice "snowballs = compounding" metaphor.

15/

Why?

Because, if acceleration flatlines, it means our snowball's *velocity* eventually grows only *linearly* with time.

Which means *radius* also grows only linearly.

And that means our snowball's mass and volume grow only cubically with time. *NOT* exponentially!

Snowball asymptotics

Acceleration = $\dot{v}$ flatlines

$\Rightarrow$ Velocity $v \sim O(t)$ [linear growth]

But $\dot{r} = \frac{k}{2\pi} \cdot \frac{v}{r}$ $\Rightarrow$ Radius $r \sim O(t)$ [again, linear]

But volume and mass are proportional to r^3

$\Rightarrow$ volume, mass $\sim O(t^3)$ [cubic growth]

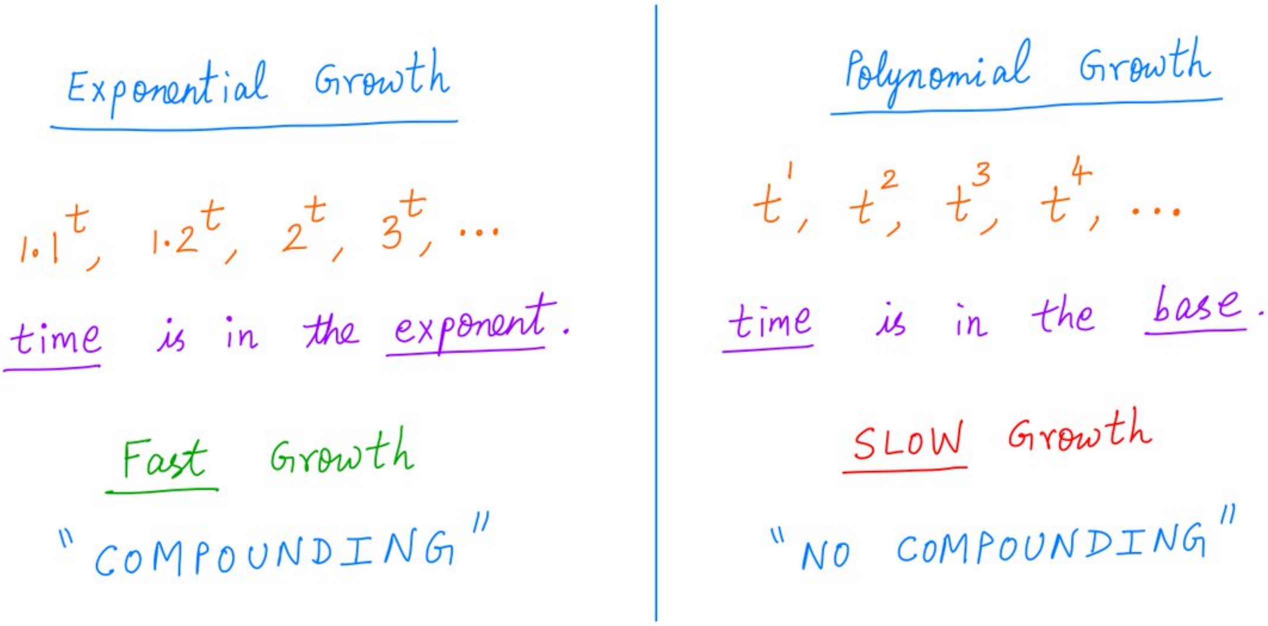
NOT Exponential!

16/

When we think of *compounding*, we think of our money growing *exponentially* with time.

Whereas the amount of snow in our snowball grows *far* more slowly -- only *polynomially* with time.

Over time, exponential growth *always* beats polynomial growth. Hands down.



17/

So here's the sad truth:

Snowballs rolling downhill grow over time (in radius, mass, volume, and speed).

But they don't *compound*.

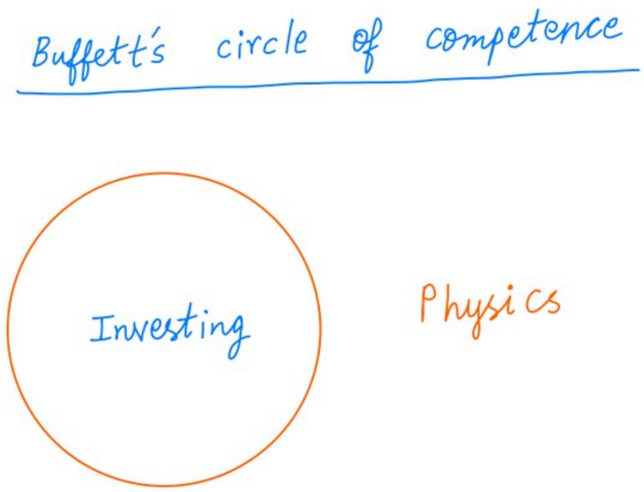
Compounding requires *exponential* growth. Snowballs only exhibit *polynomial* growth, which is much slower.

18/

Therefore, my humble request to the FinTwit community:

Please stop using snowballs as a metaphor for compounding.

Buffett was clearly straying outside his circle of competence when he used this metaphor.



19/

To learn more about differential equations -- like the snowball system we analyzed above -- I highly recommend the work of Prof. Strogatz (@stevenstrogatz).

His book, Infinite Powers, brings to life the magic of calculus and differential equations. https://t.co/4akzPM0L1K

20/

If you're somewhat more mathematically inclined, Prof. Strogatz has another gem of a book for you: Non-Linear Dynamics and Chaos. https://t.co/aRF7TH0MIg

21/

I also want to give a shout out to Grant Sanderson (@3blue1brown). I used Grant's Manim library to animate the snowball in the first tweet of this thread.

Grant makes beautiful videos explaining math concepts -- like exponential growth and pandemics: https://t.co/DhITZqbyKV

22/

If you're still with me, I cannot thank you enough!

I started writing these long form Twitter threads in April this year. It's been an amazing journey -- and I've been completely blown away by your kindness and encouragement.

Take care. Stay safe. See you in 2021!

/End