Twitter Thread by 10-K Diver

10-K Diver
@10kdiver

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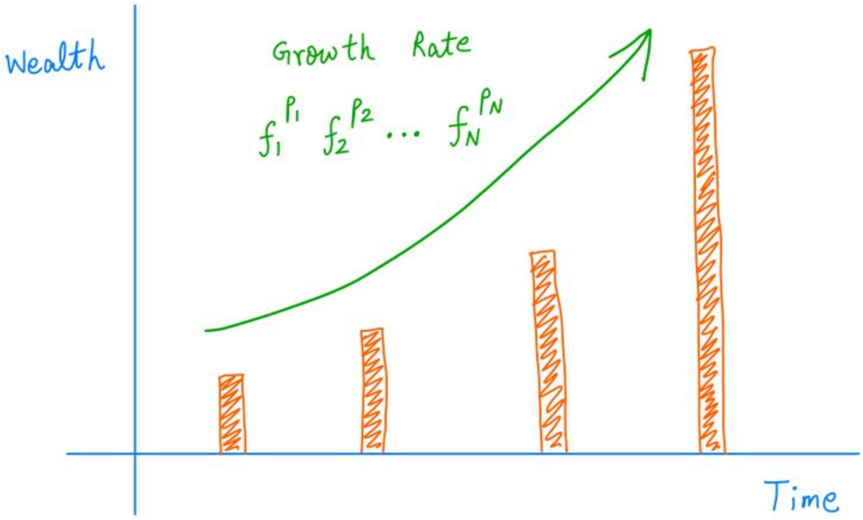
Get a cup of coffee.

In this thread, I'll help you understand Geometric Expectations.



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Imagine we have an investment opportunity.

If the investment goes well, we get to double our money.

But if it goes badly, we'll end up losing three fourths of it.

There's a 50/50 chance of either outcome.

The question: is this a good bet or not?

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A simple way to approach this question is to calculate the bet's "expectation".

For every $1 we bet, there are 2 possibilities:

1. The lucky case, where we double our money and end up with $2, and

2. The unlucky case, where we lose (3/4)'th to end up with just $0.25.

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Each of these outcomes is equally likely.

So, on *average*, every $1 we put in turns into ($2 + $0.25)/2 = $1.125.

This is a *positive* 12.5% return.

Such bets are called *positive expectation* bets. On average, we expect to *make* (rather than *lose*) money on them.

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Suppose we could take many turns betting at these odds.

In any one turn, we could of course lose money.

But because our bet has positive expectation, *over time* we expect to make money.

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In fact, as we take more and more turns, our chances of making money approach 100%, and our chances of losing money approach 0%.

In this sense, positive expectation bets are *sure things*.

They're "good" bets to make.

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For example, suppose we place 100 bets at these odds.

Let's say we bet $1 each time. So, we *put in* $100.

In this case, there's a ~93.34% chance that we'll make money.

And on average, we expect to end up with $112.50 (our +12.5% expected return).

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In the scenario above, we did not let our 100 bets *compound*.

We bet the *same* $1 at each turn.

We did not take the winnings from our first turn and roll them into our second, and so on.

But if we did that, our money would grow even faster, right? At +12.5% per turn?

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So, if we start with $1, and take 100 turns betting at these odds, rolling over our winnings from each turn to the next, we'll most likely end up with something like $1 * (1.125^100) = ~$130K, right?

That's what compounding $1 at 12.5% a hundred times gets us.

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Sadly, no.

If we roll over our winnings 100 times, our chances of making money *drop* to a tiny ~0.04%.

That is, with probability more than 99.95%, we'll end up *losing* money.

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How is this possible?

Well, let's think about what happens to our $1 as it goes through this "compounding" process 100 times.

At every "lucky" turn, our wealth gets multiplied by 2 (doubled).

But at every "unlucky" turn, it gets multiplied by 1/4 -- as (3/4)'th is lost.

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So, each unlucky turn cancels out 2 lucky turns!

So, if we are to make money overall, the lucky turns will have to outnumber the unlucky turns at least 2 to 1.

That is, we need to get lucky at least twice as often as we get unlucky -- or we'll end up losing money.

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The chances of getting lucky that often are pretty slim -- because getting lucky or unlucky is a 50/50 thing.

Over 100 turns, we're most likely to get lucky 50 times and unlucky the other 50 times.

But to make money over 100 turns, we need to get lucky at least 67 times.

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So, it all comes down to whether we "roll over" our bets or not.

Without roll over, our expected return is +12.5% per bet.

But with roll over, our expected return per bet turns *negative* -- approaching -29.3% as we take more and more turns.

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Key lesson 1: Positive expectation bets, when rolled over, can lead to *negative* expected growth rates!

Each turn may carry a positive expectation. But the compounded effect of many turns can carry a negative expectation.

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So, there are really 2 kinds of expectations.

Our normal idea of expectation -- the return we expect to make without roll over -- is called *arithmetic* expectation.

But there's also *geometric* expectation. This is the return we expect to make with roll over.

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And as our example above showed, a bet can carry positive arithmetic expectation but negative geometric expectation.

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Let's take a general bet -- and derive formulas for its arithmetic and geometric expectations.

In general, a bet can have N possible outcomes:

With probability p_1, our money gets multiplied by f_1.

With probability p_2, it gets multiplied by f_2.

And so on.

Like so:

General Bet

We have N possible outcomes:

With Probability	p_1	p_2	p_3	...	p_N
Our money gets multiplied by	f_1	f_2	f_3	...	f_N

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Here are the formulas for the arithmetic and geometric expectations of such bets, along with a couple of examples:

Arithmetic and Geometric expectations for a general bet

With Probability	p_1	p_2	p_3	...	p_N
Our money gets multiplied by	f_1	f_2	f_3	...	f_N

$$\text{Arithmetic expectation} = f_1 * p_1 + f_2 * p_2 + \dots + f_N * p_N$$

$$\text{Geometric expectation} = f_1^{p_1} * f_2^{p_2} * \dots * f_N^{p_N}$$

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Key lesson 2: Even if a bet has positive arithmetic expectation, it may be a bad idea to roll it over repeatedly.

For roll over to be a good idea, the bet must have positive *geometric* expectation.

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The famous Kelly Criterion is a strategy that transforms an *arithmetic* advantage into a *geometric* one.

The idea is: if a bet has positive arithmetic expectation but negative geometric expectation, we only roll over *part* of our winnings, not *all* of them.

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For our "double or lose three fourths" example above, Kelly would only roll over (1/6)'th of our wealth from each turn into the next turn's bet.

This achieves an expected growth rate of +10.4% per turn -- much better than negative 29.3%!

For more: https://t.co/LjuaCg3mGm

> 1) Get a cup of coffee.
>
> In this thread, I'm going to walk you through "The Kelly Criterion".
>
> — 10-K Diver (@10kdiver) May 24, 2020

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Another observation:

Whenever a bet has a non-zero chance of total loss, its geometric expectation is 0.

This is a Russian roulette type situation. Its (arithmetic) expectation may be positive, but if we keep pushing our luck and rolling over, we'll eventually blow up.

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To learn more about the origins of geometric expectations, I recommend reading Daniel Bernoulli's paper:

https://t.co/vN6yNvmk9J

Econometrica, 22, 23–36 (1954) 11

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EXPOSITION OF A NEW THEORY ON THE MEASUREMENT OF RISK[1]

BY DANIEL BERNOULLI

§1. EVER SINCE mathematicians first began to study the measurement of risk there has been general agreement on the following proposition: *Expected values are computed by multiplying each possible gain by the number of ways in which it can occur, and then dividing the sum of these products by the total number of possible cases where, in this theory, the consideration of cases which are all of the same probability is insisted upon.* If this rule be accepted, what remains to be done within the framework of this theory amounts to the enumeration of all alternatives, their breakdown into equi-probable cases and, finally, their insertion into corresponding classifications.

§2. Proper examination of the numerous demonstrations of this proposition that have come forth indicates that they all rest upon one hypothesis: *since there is no reason to assume that of two persons encountering identical risks,*[2] *either*

[1] Translated from Latin into English by Dr. Louise Sommer, The American University, Washington, D. C., from "Specimen Theoriae Novae de Mensura Sortis," *Commentarii Academiae Scientiarum Imperialis Petropolitanae*, Tomus V [*Papers of the Imperial Academy of Sciences in Petersburg*, Vol. V], 1738, pp. 175–192. Professor Karl Menger, Illinois Institute of Technology has written footnotes 4, 9, 10, and 15.

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But if that's too much for you, here's a wonderful thread by @breakingthemark that summarizes the paper's main ideas: https://t.co/fdCd3X0E9B

> I just re-read Bernoulli\u2019s 1738 paper \u201cExposition of a New Theory on the Measurement of Risk\u201d which is the foundational paper of Expected Utility Theory.
>
> It\u2019s Amazing
>
> It\u2019s so wildly different than EUT that its hard to believe this was its beginning.
>
> Let\u2019s see if you agree.
>
> — breakingthemarket (@breakingthemark) December 17, 2020

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I also very much like this article by @TaylorPearsonMe that beautifully ties together many key ideas that come into play in such repeated betting scenarios: ergodicity, antifragility, barbell strategies, the Kelly Criterion, etc.

https://t.co/FHefGhuUWf

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And speaking of ergodicity and repeated betting, Prof. Sanjay Bakshi (@Sanjay__Bakshi) has created a wonderful set of slides exploring these ideas:

https://t.co/2OsvrmvxeX

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If you're still with me, thank you very much!

Stay safe. Enjoy your weekend!

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