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## Twitter Thread by Michael Kinyon



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## Loops and their multiplication groups

## A thread in 15 parts

## (0/15)

Recall that a quasigroup (Q, \*) is a set Q with a binary operation \* such that for each a,b in Q, the equations a\*x=b and y\*a=b have unique solutions x,y. Groups are quasigroups and this property is usually one of the first things proved in elementary group theory.

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Note that we don't assume associativity of \*!

A loop is a quasigroup with an identity element. The story of why they are called loops is an interesting one and may even be true, but I will save it for another day. I am going to focus on loops in this thread.

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Natural examples of nonassociative loops:

- The nonzero octonions under multiplication

- The sphere S^7 under octonion multiplication

- I have discussed other examples previously:

https://t.co/q5LjmxHEIF

https://t.co/UPHSMwQo75

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Rethinking Vector Addition or How I Learned to Stop Worrying and Love Nonassociativity

A thread in 29 tweets

(0/28)

- Michael Kinyon (@ProfKinyon) December 1, 2020

For each x in a loop Q, define the left & right translations  $L_x$ ,  $R_x : Q \to Q$  by  $L_x(y)=xy$  and  $R_x(y)=yx$ . These mappings are permutations of Q. The composition  $L_x L_y$  of two left translations is not necessarily a left translation because Q is not necessarily associative.

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The Cayley table of a loop is a (possibly infinite) latin square with the first row and column corresponding to the identity element. You can visualize the L\_x's as being the permutations corresponding to rows and the R\_x's as being permutations corresponding to columns.

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Let Mlt(Q) ("mult Q") be the group generated by all L\_x's & R\_x's. This is called the \*multiplication group\* of Q. In other words, it's the group generated by all rows and columns of the latin square of Q.

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To help intuition, consider the case where Q is a group. Mlt(Q) puts together the left & right regular representations of Q into one group. QxQ acts on Q by  $(g,h)x = gxh^{-1} = R_h^{-1} L_g(x)$ . This gives a homomorphism from QxQ to Mlt(Q).

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Its kernel is  $K=\{(a,a)|a \text{ in } Z(Q)\}$  is a copy of the center of Q. Thus Mlt(Q) is isomorphic to QxQ/K, a complete description in this, the associative case.

Now back to the general case where Q is any loop.

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Let Inn(Q) be the stabilizer in Mlt(Q) of the identity element, that is, the subgroup of all permutations in Mlt(Q) that fix the identity element. This is called the \*inner mapping group\* of Q. When Q is a group, Inn(Q) is precisely the inner automorphism group of Q.

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Inn(Q) contains the familiar conjugations  $R_x^{-1} L_x$ . But it also contains permutations like  $L_{x^*y^{-1}} L_x L_y$ , which measure nonassociativity.

Loops can be studied via their multiplication and inner mapping groups. I will give one example of how this works.

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Some general permutation group theory: A block B of a permutation group G acting on a set X is a nonempty subset of X such that for all g in G, either gB = B (g fixes B) or gB(cap B is empty (g moves B entirely). The set {gB | g in G} forms a G-invariant partition of X.

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For a subloop N of a loop Q, the following turn out to be equivalent:

1. N is the kernel of a homomorphism,

2. N is invariant under Inn(Q),

3. N is a block of Mlt(Q).

Such an N is called a normal subloop of Q. Q is simple if it has no nontrivial normal subloops.

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A permutation group is \*primitive\* if it is transitive and has no nontrivial invariant partitions. (The trivial ones are the partition into singletons and the one part partition.)

Theorem (Albert 1941): A loop Q is simple if and only if Mlt(Q) is primitive.

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This means that, in principle, to study simple loops, we should determine which primitive groups can occur as their multiplication groups. Much is still unknown about this!

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Here is the takeaway for the main philosophical difference between semigroup theory and quasigroup theory:

Semigroups act. Quasigroups are acted upon.

That's enough for one thread. As always, thanks for reading!

(15/15)