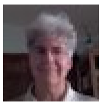
Twitter Thread by Michael Kinyon



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Loops and their multiplication groups

A thread in 15 parts

(0/15)

Recall that a quasigroup (Q,*) is a set Q with a binary operation * such that for each a,b in Q, the equations a*x=b and y*a=b have unique solutions x,y. Groups are quasigroups and this property is usually one of the first things proved in elementary group theory.

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Note that we don't assume associativity of *!

A loop is a quasigroup with an identity element. The story of why they are called loops is an interesting one and may even be true, but I will save it for another day. I am going to focus on loops in this thread.

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Natural examples of nonassociative loops:
- The nonzero octonions under multiplication
- The sphere S^7 under octonion multiplication
- I have discussed other examples previously:
https://t.co/q5LjmxHEIF
https://t.co/UPHSMwQo75
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> Rethinking Vector Addition
> or
> How I Learned to Stop Worrying and Love Nonassociativity
>
> A thread in 29 tweets

(0/28)

— Michael Kinyon (@ProfKinyon) December 1, 2020

For each x in a loop Q, define the left & right translations L_x, R_x : Q->Q by L_x(y)=xy and R_x(y)=yx. These mappings are permutations of Q. The composition L_x L_y of two left translations is not necessarily a left translation because Q is not necessarily associative.

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The Cayley table of a loop is a (possibly infinite) latin square with the first row and column corresponding to the identity element. You can visualize the L_x's as being the permutations corresponding to rows and the R_x's as being permutations corresponding to columns.

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Let Mlt(Q) ("mult Q") be the group generated by all L_x's & R_x's. This is called the *multiplication group* of Q. In other words, it's the group generated by all rows and columns of the latin square of Q.

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To help intuition, consider the case where Q is a group. Mlt(Q) puts together the left & right regular representations of Q into one group. QxQ acts on Q by (g,h)x = gxh^{-1} = R_h^{-1} L_g(x). This gives a homomorphism from QxQ to Mlt(Q).

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Its kernel is K={(a,a)|a in Z(Q)} is a copy of the center of Q. Thus Mlt(Q) is isomorphic to QxQ/K, a complete description in this, the associative case.

Now back to the general case where Q is any loop.

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Let Inn(Q) be the stabilizer in Mlt(Q) of the identity element, that is, the subgroup of all permutations in Mlt(Q) that fix the identity element. This is called the *inner mapping group* of Q. When Q is a group, Inn(Q) is precisely the inner automorphism group of Q.

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Inn(Q) contains the familiar conjugations R_x^{-1} L_x. But it also contains permutations like L_{x*y}^{-1} L_x L_y, which measure nonassociativity.

Loops can be studied via their multiplication and inner mapping groups. I will give one example of how this works.

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Some general permutation group theory: A block B of a permutation group G acting on a set X is a nonempty subset of X such that for all g in G, either gB = B (g fixes B) or gB\cap B is empty (g moves B entirely). The set {gB | g in G} forms a G-invariant partition of X.

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For a subloop N of a loop Q, the following turn out to be equivalent:
1. N is the kernel of a homomorphism,
2. N is invariant under Inn(Q),
3. N is a block of Mlt(Q).
Such an N is called a normal subloop of Q. Q is simple if it has no nontrivial normal subloops.

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A permutation group is *primitive* if it is transitive and has no nontrivial invariant partitions. (The trivial ones are the partition into singletons and the one part partition.)

Theorem (Albert 1941): A loop Q is simple if and only if Mlt(Q) is primitive.

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This means that, in principle, to study simple loops, we should determine which primitive groups can occur as their multiplication groups. Much is still unknown about this!

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Here is the takeaway for the main philosophical difference between semigroup theory and quasigroup theory:

Semigroups act.
Quasigroups are acted upon.

That's enough for one thread. As always, thanks for reading!

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