

Twitter Thread by Gro-Tsen



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@gro_tsen



OK, I may be guilty of a DoS attack attempt on mathematicians' brains here, so lest anyone waste too much precious brain time decoding this deliberately cryptic statement, let me do it for you. •1/15

$\forall x. \forall y. ((\forall z. ((z \in x) \Rightarrow (\forall t. ((t \in x) \Rightarrow (\forall u. ((u \in z) \Rightarrow (\forall v. ((v \in u) \Rightarrow (v \in y)))))) \Rightarrow (z \in y)))))) \Rightarrow (x \in y))$

— Gro-Tsen (@gro_tsen) February 12, 2021

First, as some asked, it is to be parenthesized as: “ $\forall x. \forall y. ((\forall z. ((z \in x) \Rightarrow ((\forall t. ((t \in x) \Rightarrow ((t \in z) \Rightarrow (t \in y)))))) \Rightarrow (z \in y))) \Rightarrow (\forall z. ((z \in x) \Rightarrow (z \in y))))$ ” (the convention is that ‘ \Rightarrow ’ is right-associative: “ $P \Rightarrow Q \Rightarrow R$ ” means “ $P \Rightarrow (Q \Rightarrow R)$ ”), but this doesn't clarify much. •2/15

Maybe we can make it a tad less abstruse by using guarded quantifiers (“ $\forall u \in x. (...)$ ” stands for “ $\forall u. ((u \in x) \Rightarrow (...))$ ”): it is then “ $\forall x. \forall y. ((\forall z \in x. ((\forall t \in x. ((t \in z) \Rightarrow (t \in y)))) \Rightarrow (z \in y))) \Rightarrow (\forall z \in x. (z \in y))$ ”. •3/15

Maybe a tad clearer again by writing “ $P(u)$ ” for “ $u \in y$ ” and leaving out the quantifier on y , vi■: “ $\forall x. ((\forall z \in x. ((\forall t \in x. ((t \in z) \Rightarrow P(t)))) \Rightarrow P(z))) \Rightarrow (\forall z \in x. P(z))$ ” [★]. Now it appears as an induction principle: namely, ... •4/15

... “in order to prove $P(z)$ for all $z \in x$, we can assume, when proving $P(z)$, that $P(t)$ is already known for all $t \in z \cap x$ ” (n.b.: “ $(\forall z. (Q(z) \Rightarrow P(z))) \Rightarrow (\forall z. P(z))$ ” can be read “in order to prove $P(z)$ for all z , we can assume $Q(z)$ known when proving $P(z)$ ”). •5/15

This is a principle known as “ \in -induction”: it is perhaps clearer when not guarded by x , “ $((\forall z. ((\forall t. ((t \in z) \Rightarrow P(t)))) \Rightarrow P(z)) \Rightarrow (\forall z. P(z)))$ ” [🌀], vi■ “in order to prove $P(z)$ for all z , we can assume, when proving $P(z)$, that $P(t)$ is already known for all $t \in z$ ”, ... •6/15

... but of course the guarded version [★] can be deduced from the unguarded [🌀] by applying it to $(z \in x) \Rightarrow P(z)$ in lieu of $P(z)$. This is a particular case of “well-founded induction”, and the crucial point is that the “ \in ” relation on sets is indeed well-founded: ... •7/15

... this follows (classically) from the axiom of Regularity, one of whose formulations is “ $\forall v. ((v \neq \emptyset) \Rightarrow (\exists z \in v. (z \cap v = \emptyset)))$ ” (“every set v has an element z which is disjoint from it”), or equivalently, “ $\forall v. ((\forall z \in v. (z \cap v \neq \emptyset)) \Rightarrow (v = \emptyset))$ ”: ... •8/15

... from this we can easily deduce [★] above by applying the axiom to $v := \{z \in x \mid \neg P(z)\}$, giving “ $\forall x.((\forall z \in x.(\neg P(z) \Rightarrow \exists t \in x.((t \in z) \wedge \neg P(t)))) \Rightarrow (\neg \exists z \in x. \neg P(z)))$ ” or, after a easy logical manipulations (taking contrapositives and moving negations around), ... •9/15

... “ $\forall x.((\forall z \in x.(((\forall t \in x.((t \in z) \Rightarrow P(t))) \Rightarrow P(z))) \Rightarrow (\forall z \in x. P(z)))$ ” as claimed (and, as explained earlier, the original statement then follows by applying this to “ $z \in y$ ” in lieu of $P(z)$, or, if we prefer, $v := x \blacksquare y$ in the axiom of Regularity). •10/15

(Actually, the unguarded statement “ $((\forall z.(((\forall t.((t \in z) \Rightarrow P(t))) \Rightarrow P(z))) \Rightarrow (\forall z. P(z)))$ ” [✎] also follows, but there is a little more work here, essentially constructing the transitive closure, requiring a more set-theoretic power than I have used hitherto. ... •11/15

... Conversely, this statement [✎] easily implies the axiom of Regularity by applying it to “ $z \notin v$ ” in lieu of $P(z)$. So all are, in fact, equivalent over a reasonable set theory in classical logic. ... •12/15

... But when working in intuitionistic logic it is more reasonable to work with induction principles such as [✎], which don't require moving negations around. Anyway, even classically, Regularity is mostly used for \in -induction.) •13/15

So anyway, my original statement, “ $\forall x. \forall y. ((\forall z. ((z \in x) \Rightarrow ((\forall t. ((t \in x) \Rightarrow (t \in z) \Rightarrow (t \in y)))) \Rightarrow (z \in y))) \Rightarrow (\forall z. ((z \in x) \Rightarrow (z \in y))))$ ” was essentially a (perhaps better) rephrasing of the axiom of Regularity (and equivalent to it over the rest of ZF). •14/15

It's a bit recondite, but I think, as a formula, it has a certain gnomic beauty to it — almost poetry, I might say: this is the reason I left “ $\forall z. ((z \in x) \Rightarrow (z \in y))$ ” as such and not as “ $x \subseteq y$ ”: it would spoil the rhyme. ■ •15/15