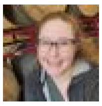
Twitter Thread by Dr. Meredith Sargent



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Ok, it's time for a #FUNctionalAnalysis thread! Let's talk about Hilbert spaces. (I hope you like linear algebra, because that's what we're

> pic.twitter.com/Q1M5zhVutP
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> — syzygay (@syzygay1) August 9, 2020

If you want to impress people, you can just say a Hilbert space is just a complete infinite dimensional inner product space and leave it at that, but let's talk about what that actually means.

When you first learn about vectors, you talk about them as arrows in space; things with a magnitude and a direction. These are elements of R^n where n is the number of dimensions of the space you care about.

You also talk about the dot product (or inner product) as a way to tell when vectors are orthogonal. (I'm purposely saying "orthogonal" instead of "perpendicular" here, but when you actually think about arrows, it's the same thing.)

As my linear algebra students are about to see, R^n is far from the only interesting vector space. A classic example is the space of polynomials of dimension less than or equal to n

We know that Taylor series can be used to represent functions that aren't polynomials as "polynomials," albeit infinitely long ones, and this is how I like to motivate infinite dimensional vector spaces.

Our vectors will be infinite sequences, and we'll need some sort of convergence requirement. To make things easy for ourselves, let's use an inner product inspired by R^n: this is called the l^2 inner product.

One thing that's nice in R^n is that we have an easy orthonormal basis. We can make "standard basis vectors" for l^2 as well.

Remember how in R^n, you can get the length of the vector by doing the dot product with itself and then taking the square root? That's the metric induced by the inner product.

In our infinite dimensional l^2, we're going to only consider vectors that have finite length.

So far, we've just made an infinite dimensional inner product space. What makes this a Hilbert space is the requirement/property that it is a "complete metric space with the metric induced by the inner product."

A metric space is complete if every Cauchy sequence converges. Think of this as saying "things that get closer and closer together also get closer and closer to a limit point." R^n is a complete metric space:

Sometimes people are disappointed to find out that all finite dimensional vector spaces "are" R^n, so hopefully I won't disappoint you too much when I say that all (infinite dimensional) Hilbert spaces "are" l^2.

More precisely, all *separable* Hilbert spaces are isomorphic to l^2. A Hilbert space is separable if it has an orthonormal basis. (All the Hilbert spaces I care about are separable.) This and the completeness requirement save us from uniqueness issues.

"Ok, so if they're all the same, what do you mean when you say 'all the Hilbert spaces?' Doesn't that mean there's only one????" Good question, hypothetical person!

"Isomorphic" here means that there's a bijective map that preserves inner products. However, this only says the spaces are the same *as Hilbert spaces.* It still allows for different sorts of objects that may have other properties.

Remember we motivated this by talking about Taylor series?

Take a (complex) holomorphic function on the complex unit disk. Being holomorphic means you can write it as a Taylor series that converges in the disk. If you add in a requirement that the coefficients are square summable, it's what we call a Hardy space function.

Even more exciting, it turns out that there's a way to define the inner product of the Hardy space as a limit of integral means, which gives you the SAME THING.

The Hilbert space structure tells you some things about this space, but because these are *functions*, there are lots of other properties that come with that. (Factorization, multiplier algebras, boundary behavior, etc)

There are also other ways to define inner products for spaces of functions. These function spaces will be isomorphic (as Hilbert spaces) to l^2, but the function theoretic properties will vary.

One way to "tell these apart" is to look at whether or not it's a "Reproducing Kernel Hilbert Space" and to look at what the reproducing kernel is.

So next time on #FUNctionalAnalysis, we'll talk about Reproducing Kernel Hilbert Spaces! Get psyched!

Further reading:
An Introduction to Hilbert Space by Nicholas Young
Ryan Tully-Doyle's Hilbert space interactive text: https://t.co/ulX5VK0tbk