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I just re-read Bernoulli's 1738 paper "Exposition of a New Theory on the Measurement of Risk" which is the foundational paper of Expected Utility Theory.

It's Amazing

It's so wildly different than EUT that its hard to believe this was its beginning.

Let's see if you agree.

The paper isn't about utility. It's about expected value.

Bernoulli used the utility concept to get the reader to abandon the traditional view of expected value(arithmetic average), and then used it to derive the equation for valuing risk.

The final equation doesn't use utility

He starts out the paper identifying that tradition evaluation of risk come from expected values, which are calculated with the arithmetic average.

Notice the rule here in italics is about expected values.

EXPOSITION OF A NEW THEORY ON THE MEASUREMENT OF RISK[1]

By Daniel Bernoulli

§1. Ever since mathematicians first began to study the measurement of risk there has been general agreement on the following proposition: *Expected values are computed by multiplying each possible gain by the number of ways in which it can occur, and then dividing the sum of these products by the total number of possible cases where, in this theory, the consideration of cases which are all of the same probability is insisted upon.* If this rule be accepted, what remains to be done within the framework of this theory amounts to the enumeration of all alternatives, their breakdown into equi-probable cases and, finally, their insertion into corresponding classifications.

Then points out that this average ignores anything about the specific financial circumstances of the participant. It's not a risk aversion point, but that a poor person values money differently than a rich person.

by the highest judges established by public authority. But really there is here no need for judgment but of deliberation, i.e., rules would be set up whereby anyone could estimate his prospects from any risky undertaking in light of one's specific financial circumstances.

§3. To make this clear it is perhaps advisable to consider the following example: Somehow a very poor fellow obtains a lottery ticket that will yield with equal probability either nothing or twenty thousand ducats. Will this man evaluate his chance of winning at ten thousand ducats? Would he not be ill-advised to sell this lottery ticket for nine thousand ducats? To me it seems that the answer is in the negative. On the other hand I am inclined to believe that a

Therefore the rule for determining expected values must not be correct. An item is valued by its usefulness (utility) therefore the expected value must also reflect the usefulness it yields.

This was originally Latin, so not sure of the translation, but I love the word yields.

same rule to evaluate the gamble. The rule established in §1 must, therefore, be discarded. But anyone who considers the problem with perspicacity and interest will ascertain that the concept of *value* which we have used in this rule may be defined in a way which renders the entire procedure universally acceptable without reservation. To do this the determination of the *value* of an item must not be based on its *price*, but rather on the *utility* it yields. The price of the item is dependent only on the thing itself and is equal for everyone; the utility, however, is dependent on the particular circumstances of the person making the estimate. Thus there is no doubt that a gain of one thousand ducats is more significant to a pauper than to a rich man though both gain the same amount.

I also love that Bernoulli noticed a disconnect between using the arithmetic average as the expected value and how the world really works.

Instead of calling people flawed for not following it, he chose to call the theory flawed instead.

The idea of the arithmetic average being expected and the foundation of rational decisions was pretty well entrenched at the time.

So he has to use the concept of utility here to break people of this belief that the arithmetic was expected.

Essentially explaining that people should have different usefulness for the money.

He runs through various examples to convince people that the arithmetic average can't possibly describe the true risk, outlining some extreme cases to emphasize the flaw in the arithmetic average.

After he does that though, he returns to normal and say, most of these examples were unusual. We should focus instead on what's typical, which he says is that people find usefulness from money in proportion to their wealth.

As in usefulness = money / wealth.

Though innumerable examples of this kind may be constructed, they represent exceedingly rare exceptions. We shall, therefore, do better to consider what usually happens, and in order to perceive the problem more correctly we shall assume that there is an imperceptibly small growth in the individual's wealth which proceeds continuously by infinitesimal increments. Now it is highly probable that *any increase in wealth, no matter how insignificant, will always result in an increase in utility which is inversely proportionate to the quantity of goods already possessed*. To explain this hypothesis it is necessary to define what is meant by the *quantity of goods*. By this expression I mean to connote food, clothing, all things which add to the conveniences of life, and even to luxury—anything that can contribute to the adequate satisfaction of any

Then he runs off on a tangent about income, and then returns to the previous point to double down on it, repeating it nearly identically.

Money/wealth is simply growth.

So he’s saying we derive usefulness from growth, not absolute price.

This is his foundational point.

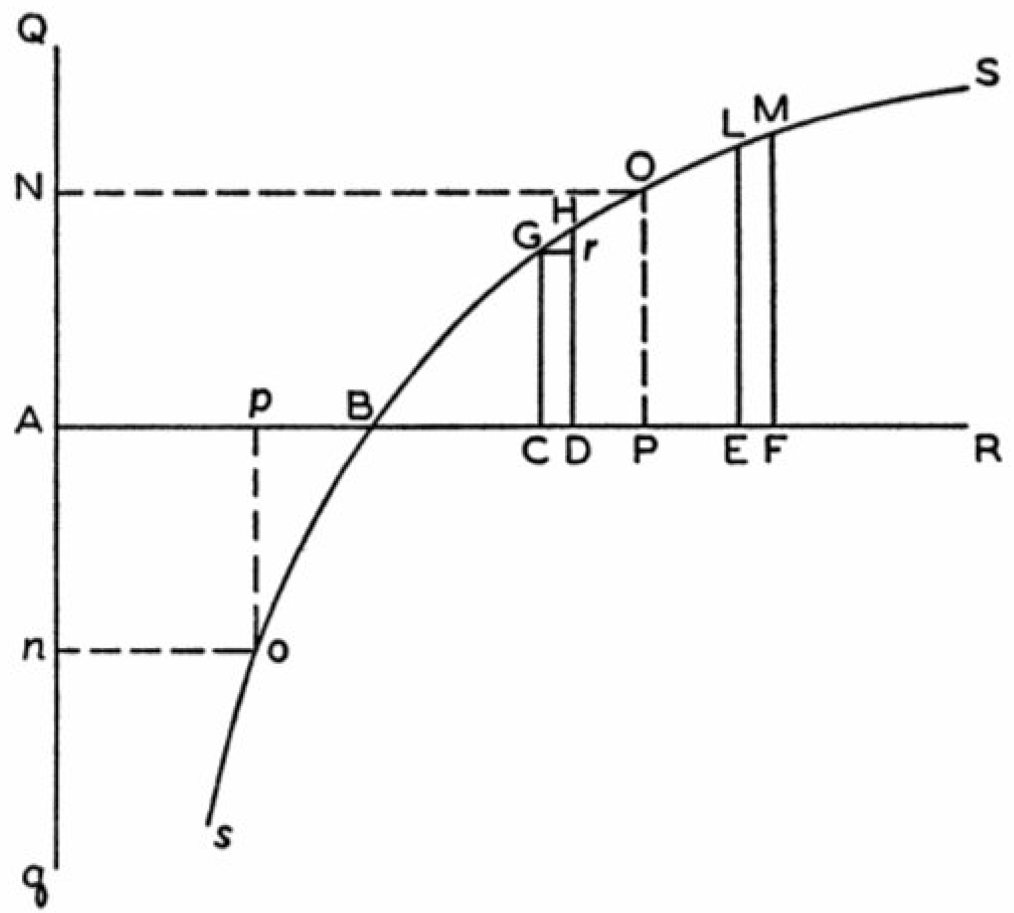
> §6. Having stated this definition, I return to the statement made in the previous paragraph which maintained that, in the absence of the unusual, the *utility resulting from any small increase in wealth will be inversely proportionate to the quantity of goods previously possessed*. Considering the nature of man, it seems to me that the foregoing hypothesis is apt to be valid for many people to whom this sort of comparison can be applied. Only a few do not spend their entire yearly incomes. But, if among these, one has a fortune worth a hundred thousand ducats and another a fortune worth the same number of semi-ducats and if the former receives from it a yearly income of five thousand ducats while the latter obtains the same number of semi-ducats it is quite clear that to the former a ducat has exactly the same significance as a semi-ducat to the latter, and that, therefore, the gain of one ducat will have to the former no higher value than the gain of a semi-ducat to the latter. Accordingly, if each makes a gain of one ducat the latter receives twice as much utility from it, having been enriched by two semi-ducats. This argument applies to many other cases which, therefore, need not

To give you an analogy from today’s world, Imagine Bernoulli surrounded by a everyone quoting Dow moves in points.

This drove him nuts and his response was, “nobody cares about points, you should quote it in percent gain.”

So now that the hypothesis is clear--usefulness comes from growth, not absolute gains--he begins to derive the formula. He starts out with a generic “utility curve”. It’s clearly a logarithm, but thats not stated or derived yet. He’s just dropping logical constraints now.

Since, however, in special cases things can conceivably occur otherwise, I shall first deal with the most general case and then develop our special hypothesis in order thereby to satisfy everyone.



§7. Therefore, let AB represent the quantity of goods initially possessed. Then after extending AB, a curve $BGLS$ must be constructed, whose ordinates CG, DH, EL, FM, etc., designate *utilities* corresponding to the abscissas BC, BD, BE, BF, etc., designating gains in wealth. Further, let m, n, p, q, etc., be the numbers which indicate the number of ways in which gains in wealth BC, BD, BE, BF [misprinted in the original as CF], etc., can occur. Then (in accord with §4) the *moral* expectation of the risky proposition referred to is given by:

$$PO = \frac{m.CG + n.DH + p.EL + q.FM + \cdots}{m + n + p + q + \cdots}$$

Now, if we erect AQ perpendicular to AR, and on it measure off $AN = PO$, the straight line $NO - AB$ represents the gain which may properly be expected, or the value of the risky proposition in question. If we wish, further, to know how

The x axis is real world wealth. The Y axis is utility. B is the current wealth.

Bernoulli averages this mythical utility. The PO line is this average, from the utility of 4 possible outcomes GC, DH, EL, and FM. This is where EUT starts comparing things

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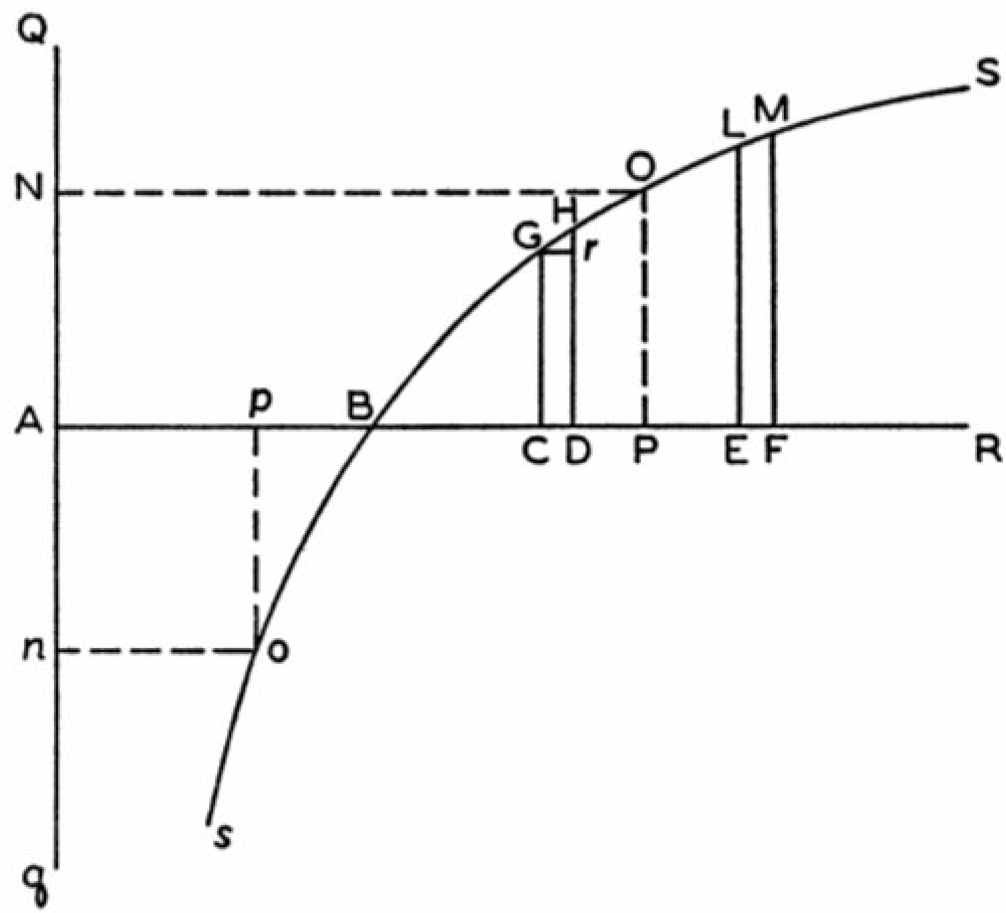

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But Bernoulli aims to then to translate P back to real world wealth at O.

He went into utility space did some math, and then left utility, returning to the real world.

Next he calculates the EXPECTED GAIN by subtracting this value O from the original value B.

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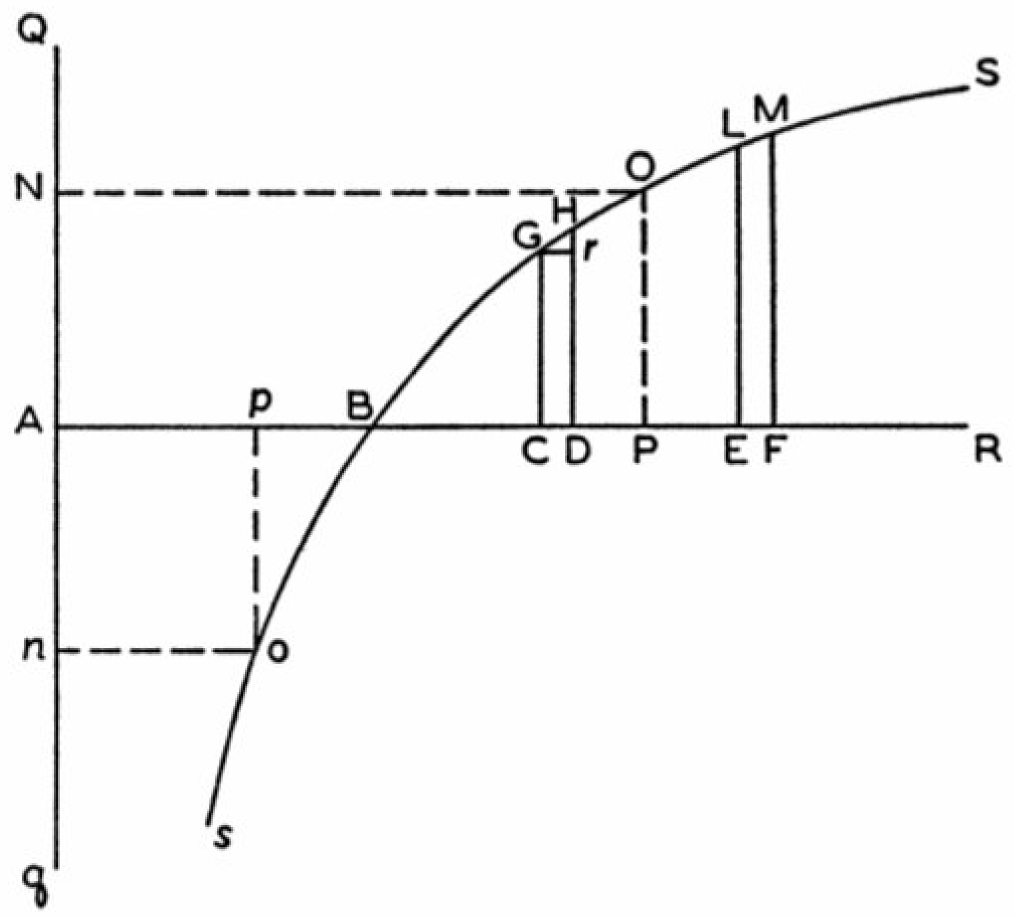

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He says to go into the utility function (which he hasn't made clear is a logarithm yet), average the results, then leave utility to calculate a new expected value!

Bernoulli, doesn't care about utility, he's only using it calculate the correct expected value in our world!

Bernoulli then maps the current world's beliefs of expectation into his chart method. Remember, he hasn't yet said it's a logarithm.

So he shows that in the arithmetic view of expected values, the curve isn't a curve but a straight line. He still works back to real values(BP).

COROLLARY I

§8. Until now scientists have usually rested their hypothesis on the assumption that all gains must be evaluated exclusively in terms of themselves, i.e., on the basis of their intrinsic qualities, and that these gains will always produce a *utility* directly proportionate to the gain. On this hypothesis the curve BS becomes a straight line. Now if we again have:

$$PO = \frac{m.CG + n.DH + p.EL + q.FM + \cdots}{m + n + p + q + \cdots},$$

and if, on both sides, the respective factors are introduced it follows that:

$$BP = \frac{m.BC + n.BD + p.BE + q.BF + \cdots}{m + n + p + q + \cdots},$$

which is in conformity with the usually accepted rule.

The paper next returns back the key hypnosis:

Usefulness of gains is inversely proportional to current wealth

and derives the equation for this curve.

This is where he shows his hypothesis leads to a logarithmic utility curve.

$b \log \frac{x}{\alpha}$. The curve sBS is therefore a logarithmic curve, the subtangent[4] of which is everywhere b and whose asymptote is Qq.

He then takes this logarithm curve and goes back to his original chart method of expected value from utility, combines them, and simplifies.

Producing his final equation and ultimate solution for a new expected value.

It is a geometric average.

§11. If we now compare this result with what has been said in paragraph 7, it will appear that: $PO = b \log AP/AB$, $CG = b \log AC/AB$, $DH = b \log AD/AB$ and so on; but since we have

$$PO = \frac{m.CG + n.DH + p.EL + q.FM + \cdots}{m + n + p + q + \cdots}$$

it follows that

$$b \log \frac{AP}{AB} = \left(mb \log \frac{AC}{AB} + nb \log \frac{AD}{AB} + pb \log \frac{AE}{AB} + qb \log \frac{AF}{AB} + \cdots\right) : (m + n + p + q + \cdots)$$

and therefore

$$AP = (AC^m . AD^n . AE^p . AF^q \cdot \cdots)^{1/m+n+p+q+\cdots}$$

and if we subtract AB from this, the remaining magnitude, BP, will represent the value of the risky proposition in question.

Its eye opening that Bernoulli's final equation doesn't include utility at all. Its been simplified out. Furthermore, the equation is meant to find the value of the risky proposition. It's supposed to convey the risk's worth in actual real dollars, not an imaginary utility value.

It's also eye opening that his ultimate rule compares the geometric return of the potential outcomes to the original wealth.

He has replaced the arithmetic return with the geometric return. The geometric average indicates the value of the risky proposition.

§12. Thus the preceding paragraph suggests the following rule: *Any gain must be added to the fortune previously possessed, then this sum must be raised to the power given by the number of possible ways in which the gain may be obtained; these terms should then be multiplied together. Then of this product a root must be extracted the degree of which is given by the number of all possible cases, and finally the value of the initial possessions must be subtracted therefrom; what then remains indicates the value of the risky proposition in question.* This principle is essential for the measurement of the value of risky propositions in various cases. I would elaborate it into a complete theory as has been done with the traditional analysis, were it not that, despite its usefulness and originality, previous obligations do not permit me to undertake this task. I shall therefore, at this time, mention only the more significant points among those which have at first glance occurred to me.

Bernoulli only used utility to convince people they were valuing risk wrong, and then as necessary to derive the final solution.

But utility isn't part of his solution, it was just a tool to get there.

The solution is the geometric average.

The paper then tackles examples and shows some interesting finding from employing the geometric average to value risk.

First, he shows that "fair bet" of risk 50 to gain 50 has an expected loss of 13.

Start with 100. Sqrt(50 x 150) = 86.6. 100-86.6 =13.4

> everyone. Let us assume that of two players, both possessing one hundred ducats, each puts up half this sum as a stake in a game that offers the same probabilities to both players. Under this assumption each will then have fifty ducats plus the expectation of winning yet one hundred ducats more. However, the sum of the values of these two items amounts, by the rule of §12, to only $(50^{1} . 150^{1})^{\frac{1}{2}}$ or $\sqrt{50 . 150}$, i.e., less than eighty-seven ducats, so that, though the game be played under perfectly equal conditions for both, either will suffer an expected loss of more than thirteen ducats. We must strongly emphasize this truth, although it be self evident: the imprudence of a gambler will be the greater the larger the part of his fortune which he exposes to a game of chance. For this purpose we shall modify the previous example by assuming that one of the gamblers, before putting up his fifty ducat stake possessed two hundred ducats. This gambler suffers an expected loss of $200 - \sqrt{150 . 250}$, which is not much greater than six ducats.

I love that he then calls those using the arithmetic return to evaluate fair bets irrational. I wonder if most economists realize one of the foundation documents of their professions says they are the irrational ones, not their subjects that they classify as risk averse.

§14. Since, therefore, everyone who bets any part of his fortune, however small, on a mathematically fair game of chance acts irrationally, it may be of interest to inquire how great an advantage the gambler must enjoy over his opponent in order to avoid any expected loss. Let us again consider a game which is as simple as possible, defined by two equiprobable outcomes one of which is favorable and the other unfavorable. Let us take a to be the gain to be won in case of a favorable outcome, and x to be the stake which is lost in the unfavorable case. If the initial quantity of goods possessed is α we have $AB = \alpha$; $BP = a$; $PO = b \log \frac{\alpha + a}{\alpha}$ (see §10), and since (by §7) $po = PO$ it follows by the nature of a logarithmic curve that $Bp = \frac{\alpha a}{\alpha + a}$. Since however Bp represents the stake x, we have $x = \frac{\alpha a}{\alpha + a}$ a magnitude which is always smaller than a, the expected gain. It also follows from this that a man who risks his entire fortune acts like a simpleton, however great may be the possible gain. No one will have difficulty in being persuaded of this if he has carefully examined our definitions given above. Moreover, this result sheds light on a statement which is universally accepted in practice: it may be reasonable for some individuals to invest in a doubtful enterprise and yet be unreasonable for others to do so.

After the irrationality jab, he solves for the properties of a “fair game” for his new view of the expected gain and loss. It takes a bit or rearranging, but this formula matches the Kelly Criterion discovered 200 years later.

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Then he tackles a much more complicated problem of shipping insurance.

His solution is entirely about comparing geometric averages. No utility.

He calls the averages the expectation. This is a “size of the bet calculation”, once again similar to Kelly.

§15. The procedure customarily employed by merchants in the insurance of commodities transported by sea seems to merit special attention. This may again be explained by an example. Suppose Caius,[5] a Petersburg merchant, has pur-

[5] Caius is a Roman name, used here in the sense of our "Mr. Jones." Caius is the older form; in the later Roman period it was spelled "Gaius." [Translator]

chased commodities in Amsterdam which he could sell for ten thousand rubles if he had them in Petersburg. He therefore orders them to be shipped there by sea, but is in doubt whether or not to insure them. He is well aware of the fact that at this time of year of one hundred ships which sail from Amsterdam to Petersburg, five are usually lost. However, there is no insurance available below the price of eight hundred rubles a cargo, an amount which he considers outrageously high. The question is, therefore, how much wealth must Caius possess apart from the goods under consideration in order that it be sensible for him to abstain from insuring them? If x represents his fortune, then this together with the value of the expectation of the safe arrival of his goods is given by $\sqrt[100]{(x + 10000)^{95}x^5} = \sqrt[20]{(x + 10000)^{19}x}$ in case he abstains. With insurance he will have a certain fortune of $x + 9200$. Equating these two magnitudes we get: $(x + 10000)^{19}x = (x + 9200)^{20}$ or, approximately, $x = 5043$. If, therefore, Caius, apart from the expectation of receiving his commodities, possesses an amount greater than 5043 rubles he will be right in not buying insurance. If, on the contrary, his wealth is less than this amount he should insure his cargo. And if the question be asked "What minimum fortune should be possessed by the man who offers to provide this insurance in order for him to be rational in doing so?" We must answer thus: let y be his fortune, then

$$\sqrt[20]{(y + 800)^{19} . (y - 9200)} = y$$

or approximately, $y = 14243$, a figure which is obtained from the foregoing without additional calculation. A man less wealthy than this would be foolish to provide the surety, but it makes sense for a wealthier man to do so. From this it is clear that the introduction of this sort of insurance has been so useful since it offers advantages to all persons concerned. Similarly, had Caius been able to obtain the insurance for six hundred rubles he would have been unwise to refuse it if he possessed less than 20478 rubles, but he would have acted much too cautiously had he insured his commodities at this rate when his fortune was greater than this amount. On the other hand a man would act unadvisedly if he were to offer to sponsor this insurance for six hundred rubles when he himself possesses less than 29878 rubles. However, he would be well advised to do so if he possessed more than that amount. But no one, however rich, would be manag-

Next he explores the usefulness of diversification, calculating the benefit of diversification with the geometric average, and calling the calculation his expectation.

300 years go Bernoulli proved, and quantified, the diversification benefit.

§16. Another rule which may prove useful can be derived from our theory. This is the rule that it is advisable to divide goods which are exposed to some danger into several portions rather than to risk them all together. Again I shall explain this more precisely by an example. Sempronius owns goods at home worth a total of 4000 ducats and in addition possesses 8000 ducats worth of commodities in foreign countries from where they can only be transported by sea. However, our daily experience teaches us that of ten ships one perishes. Under these conditions I maintain that if Sempronius trusted all his 8000 ducats of goods to one ship his expectation of the commodities is worth 6751 ducats. That is

$$\sqrt[10]{12000^9 . 4000^1} - 4000.$$

If, however, he were to trust equal portions of these commodities to two ships the value of his expectation would be

$$\sqrt[100]{12000^{81} . 8000^{18} . 4000} - 4000, \text{ i.e., } 7033 \text{ ducats.}$$

In this way the value of Sempronius' prospects of success will grow more favorable the smaller the proportion committed to each ship. However, his expectation will never rise in value above 7200 ducats. This counsel will be equally serviceable for those who invest their fortunes in foreign bills of exchange and other hazardous enterprises.

Finally to the St. Petersburg Paradox. Funny how the key puzzle is last, and could be dropped entirely with no effect.

He of course solves the paradox with the geometric average, and interestingly points out the value of something you own is different than something to purchase

§18. The number of cases to be considered here is infinite: in one half of the cases the game will end at the first throw, in one quarter of the cases it will conclude at the second, in an eighth part of the cases with the third, in a sixteenth part with the fourth, and so on.[10] If we designate the number of cases through infinity by N it is clear that there are $\frac{1}{2}N$ cases in which Paul gains one ducat, $\frac{1}{4}N$ cases in which he gains two ducats, $\frac{1}{8}N$ in which he gains four, $\frac{1}{16}N$ in which he gains eight, and so on, ad infinitum. Let us represent Paul's fortune by α; the proposition in question will then be worth

$$\sqrt[N]{(\alpha+1)^{N/2}.(\alpha+2)^{N/4}.(\alpha+4)^{N/8}.(\alpha+8)^{N/16}\cdots} - \alpha$$
$$= \sqrt{(\alpha+1)}.\sqrt[4]{(\alpha+2)}.\sqrt[8]{(\alpha+4)}.\sqrt[16]{(\alpha+8)}\cdots - \alpha.$$

§19. From this formula which evaluates Paul's prospective gain it follows that this value will increase with the size of Paul's fortune and will never attain an infinite value unless Paul's wealth simultaneously becomes infinite. In addition we obtain the following corollaries. If Paul owned nothing at all the value of his expectation would be

$$\sqrt[2]{1}.\sqrt[4]{2}.\sqrt[8]{4}.\sqrt[16]{8}\cdots$$

which amounts to two ducats, precisely. If he owned ten ducats his opportunity would be worth approximately three ducats; it would be worth approximately four if his wealth were one hundred, and six if he possessed one thousand. From this we can easily see what a tremendous fortune a man must own for it to make sense for him to purchase Paul's opportunity for twenty ducats. The amount which the buyer ought to pay for this proposition differs somewhat from the amount it would be worth to him were it already in his possession. Since, however, this difference is exceedingly small if α (Paul's fortune) is great,

This paper is just spectacular, and so much of it is ignored. Its essential point is:

The geometric average, not the arithmetic average, is the expected value and decisions involving risk should be judged from this value.

So why do economists use utility, when Bernoulli condensed it out of his equation and didn't use it to solve problems? Why do they measure risk aversion from the arithmetic average?

Why does everyone continue to use the arithmetic average as the expected value?

Will it be 3 more centuries until everyone realizes to expect the geometric average?

Full is paper here.
Bernoulli.pdf (https://t.co/jgjsFIfOFL)

Read it, and then read economist description of what they think he said. See if you agree Bernoulli prescribed a different solution